Differential Equations

Please work out all five problems.

1.(20%) Let M, C, and K be real symmetric positive-definite $n \times n$ matrices. Recall that a matrix A is called positive-definite if $x^T Ax > 0$ for all $x \neq 0$. Show that any solution to the second order system

$$M\ddot{y} + C\dot{y} + Ky = 0$$

tends to zero as $t \to \infty$.

2.(20%) Assume that $f(s) : \mathbb{R} \to \mathbb{R}$ is a C^1 function with $f(1) = 0$. Let $(x(t), y(t))$ be a solution of the following system of equations

$$\begin{cases} x'(t) = x(t)f(x(t)^2 + y(t)^2) - y(t), & x(0) = a, \quad 0 < a < 1, \\ y'(t) = y(t)f(x(t)^2 + y(t)^2) + x(t), & y(0) = 0. \end{cases}$$

Then show that $x(t)$ and $y(t)$ are bounded functions on $\mathbb{R}$.

3.(20%) Using the method of Frobenius to find two linearly independent solutions of

$$L[y] = t^2y'' + (t^2 + t)y' - y = 0, \quad t > 0.$$

4.(20%) Show that all solutions $x(t), y(t)$ of

$$\begin{cases} x'(t) = x^2 + y \sin x, \\ y'(t) = -1 + xy + \cos y \end{cases}$$

which start in the first quadrant $(x > 0, y > 0)$ must remain there for all time.

5.(20%) Find all eigenvalues and eigenvectors of the boundary value problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, \quad y(1) + y'(1) = 0.$$

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