

Work out all problems with details to get full credits.

- (1) (a) Let V be \mathbb{R}^2 , with the standard inner product. If U is an unitary operator on V , show that the matrix of U in the standard ordered basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$ is either (10%)

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

for some real $\theta, 0 \leq \theta \leq 2\pi$.

- (b) Let ϕ be a fixed real number, and let $B = \{\alpha_1, \alpha_2\}$ be the orthonormal basis obtained by rotating $\{e_1, e_2\}$ through the angle ϕ . Let θ be another real number. Find the matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ in the ordered basis B . (10%)

- (2) (a) Let A be a square matrix satisfying $A^2 = A$. Find the distinct eigenvalues of A . (10%)

- (b) Let A be a square matrix of rank k . Show that the number of distinct nonzero eigenvalues of A is less than or equal to k . (10%)

- (c) Consider the $n \times n$ matrix (10%)

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Find the eigenvalues of A . (Hint. Try some eigenvectors of A . What is the rank of A ?)

- (3) Let T be the linear operator on \mathbb{R}^2 , the matrix of which in the standard ordered basis is $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Let W_1 be the subspace of \mathbb{R}^2 spanned by the vector $e_1 = (1, 0)$.

- (a) Show that W_1 is invariant under T . (5%)

- (b) Prove that there is no subspace W_2 which is invariant under T and which is complementary to W_1 (i.e., no subspace W_2 satisfying $\mathbb{R}^2 = W_1 \oplus W_2$.) (5%)

Let $P_2(\mathbb{R})$ be the vector space of all polynomial functions p from \mathbb{R} to \mathbb{R} which has degree 2 or less (so $p(x) = c_0 + c_1x + c_2x^2$ for some $c_i \in \mathbb{R}$). Also, denote by $M_n(\mathbb{R})$ the set of all $n \times n$ matrices over \mathbb{R} .

- (4) Define three linear functionals on $P_2(\mathbb{R})$ by (10%)

$$f_1(p) = \int_0^1 p(x) dx, \quad f_2(p) = \int_0^2 p(x) dx, \quad f_3(p) = \int_0^{-1} p(x) dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis of the dual space $P_2(\mathbb{R})^*$. (Hint. Find the dual basis of it in $P_2(\mathbb{R})$.)

- (5) Compute the minimal polynomial for each of the following linear operators.

- (a) $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$, where $T(p) = p' + 2p$; (5%)

- (b) $T : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$, where $T(A) = A^t$, the transpose of A . (5%)

- (6) Let V be a finite dimensional vector space over \mathbb{C} , and A and B two commuting linear transformations of V . Show that there is at least one common eigenvector for A and B . (10%)

(背面仍有題目,請繼續作答)

- (7) Let V be the linear subspace spanned by $(1, 0, 2, 3), (2, 3, 0, 1), (3, 1, 2, 0) \in \mathbb{R}^4$. Find an orthonormal basis of V . (10%)