Advanced Calculus Entrance Exam

Spring 2001

(1) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $f(x,y) = (e^x \cos y, e^x \sin y)$.

(i) Show that $Df(x,y)$ is invertible at every point of $\mathbb{R}^2$. 5%

(ii) Show that f is not one-to-one. 5%

(iii) Does (i) and (ii) contradict the *Inverse Function Theorem*? Why? 5%

(2) Let R be a bounded closed set in $\mathbb{R}^2$ and C be the smooth boundary curve. *The Green's second theorem* states:

$$\iint_R (u\Delta w - w\Delta u)dxdy = \int_C \Big(u\frac{dw}{dn} - w\frac{du}{dn}\Big)ds.$$

where w, u are both C^2 functions on R.

(i) How should you define the orientation of R, C and $\vec{n}$ to make the formula correct? 5%

(ii) How can you interpret this theorem as a formula for *Integration by parts*? 5%

(iii) Let us define $< f, g >_R = \iint_R f(x,y)g(x,y)dxdy$ for any $f, g \in \Omega$ where Ω is the vector space of all C^∞ functions whose directional derivatives in the direction of normal at all points of C vanish. A linear operator $L : \Omega \to \Omega$ is said to be *self-adjoint* if it satisfies $< Lf, g >_R = < f, Lg >_R$. Show that the Laplace Operator Δ is self-adjoint. 5%

(3) Let f be a continuous function of two variables (t,x) defined for $t \geq a$ and x in some compact set $S \subset \mathbb{R}$. Assume that the integral

$$\int_a^\infty f(t,x)dt = \lim_{B\to\infty} \int_a^B f(t,x)dt$$

converges uniformly for $x \in S$.

(i) Show that $g(x) = \int_a^\infty f(t,x)dt$ is continuous for $x \in S$. 10%

(ii) Does $\int_0^\infty xe^{-tx}dt$ converge uniformly for $x \in [0,1]$? Verify your answer. 10%

(4) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an invertible linear mapping and $\mathbf{B}_r$ be an n-dimensional ball centered at $\mathbf{0}$ with radius r. Compute 10%

$$\lim_{r\to\infty} \int_{T^{-1}(\mathbf{B}_r)} e^{-<T\mathrm{y},T\mathrm{y}>}d\mathrm{y}.$$

(5) Let $u = (u_1, u_2, ..., u_n)^t \in \mathbb{R}^n$; $f_j(u), j = 1, 2, \ldots, q$ are continuously differentiable on $\mathbb{R}^n$. Consider 10%

$$L_p(u,\lambda) = \sum_{j=1}^q \lambda_j f_j(u) - (1/p)\sum_{j=1}^q \lambda_j \ln \lambda_j,$$

where $p > 0$ and $\lambda \in \Delta = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q) \geq 0 | \sum_{j=1}^q \lambda_j = 1\}$. Show that, for each fixed $p > 0$ and $u \in \mathbb{R}^n$, there is a unique optimal solution:

$$\lambda_j^*(u,p) = \exp\big(pf_j(u)\big) / \sum_{j=1}^q \exp\big(pf_j(u)\big),\ j = 1, 2, \ldots, q.$$

that maximizes $L_p(u,\lambda)$ over $\lambda \in \Delta$.

(6) Let l be a positive integer. Define

$$\Phi(\theta) = \frac{1}{l}\sum_{m=1}^l \cos\theta_m,$$

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where $\theta = (\theta_1, \theta_2, \ldots, \theta_l)$. Further define

$$p(n, x, y) = \frac{1}{(2\pi)^l} \int_Q e^{i<\theta,(x-y)>} \Phi^n(\theta) d\theta,$$

where $x, y \in \mathbb{R}^l$, $Q = \{\theta | -\pi \le \theta_m \le \pi, \forall m = 1, 2, \ldots, l\}$ and $<\theta, (x-y)>$ is the usual inner product in $\mathbb{R}^l$. Consider

$$g(x, y) = \sum_{n=0}^{\infty} p(n, x, y).$$

(i) Show that $g(x, y) \le \frac{1}{(2\pi)^l} \int_Q \frac{d\theta}{1 - |\Phi(\theta)|}$. 10%

(ii) Show that, there exists a neighborhood U of the point $\theta = (0, 0, \ldots, 0)$ in which 10%

$$\int_U \frac{d\theta}{1 - |\Phi(\theta)|} < \int_U \frac{4l d\theta}{\theta_1^2 + \theta_2^2 + \cdots + \theta_l^2}.$$

(Hint: Use the first two terms of Taylor's expansion for each $\cos\theta_j$.)

(iii) Use (ii) to show that $g(x, y) < \infty$ for $l \ge 3$. (You may first try $l = 3$ using Spherical Coordinates) 10%