1. 直接寫下以下各常微分方程式之解$y(x)$. (10%)

(a) $\frac{d}{dx}y = y,\ y(0) = 1.$

(b) $\frac{d^2}{dx^2}y = -y,\ y(0) = 1, y'(0) = 0.$

(c) $\frac{d^2}{dx^2}y = y,\ y(0) = 0, y'(0) = 2.$

(d) $\frac{d}{dx}y = x,\ y(0) = 0.$

2. 求以下各常微分方程式之所有解. (20%)

(a) $\frac{d^2}{dx^2}y = -4y,\ y(0) = 1.$

(b) $\frac{d}{dx}y = y + x.$

(c) $\frac{dy}{dx} = xy.$

(d) $2xy\frac{d}{dx}y = x^2 + y^2.$

3. 方陣$A = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$且$A = Q\Lambda Q^T$,其中$Q = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$

(a) 解一階聯立方程式 (10%)

$$\frac{d}{dx}Y(x) = AY + x\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) 解二階聯立方程式 (10%)

$$\frac{d^2}{dx^2}Y(x) = AY, \quad Y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \frac{d}{dx}Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

4. $\frac{d}{dx^2}y + \lambda\frac{d}{dx}y + Py = 0,\ y(0) = 1,\ \frac{d}{dx}y(0) = 0$,其中$\lambda, P > 0$,求令$y(x)$在區間$(0,\infty)$不爲零之條件($\lambda$與$P$之關係). (10%)

5. $\frac{d^2}{dx^2}y + \omega^2 y = \sin kx,\ y(0) = 0,\ \frac{d}{dx}y(0) = 0.$ (10%)

(a) 求以上初始值問題之解. (設$\omega \neq k$)

(b) 求k趨近於ω時之解. ($\lim_{k\to\omega} y(x)$)

6. 求以下初始值問題之power series解. (10%)

$$(1 - x^2)\frac{d^2}{dx^2}y + 2x\frac{d}{dx}y + 6y = 0, \quad y(0) = 0, \quad \frac{d}{dx}y(0) = 1.$$

7. 函數$u(x,t)$滿足wave equation

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{\partial^2}{\partial t^2}u(x,t),$$

邊界條件$u(0,t) = 0$與$u(2\pi,t) = 0$,初始條件$u(x,0) = \sin x + \sin 2x$.
求$u(x,t)$之解. (20%)