1. (10%) Construct the second Lagrange interpolating polynomial for $f(x) = 2/x$, using the nodes $x_0 = 1$, $x_1 = 1.5$, and $x_2 = 4$.

2. (10%) A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x < 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find b, c, and d.

3. (10%) Approximate the following integral using Gaussian quadrature with $n = 2$:

$$\int_1^{1.5} x \ln x \, dx.$$

4. (12%) Find a factorization of the form $A = LL^t$ for the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

5. (10%) Find the first two iterations of the Jacobi method, using $\mathbf{x}^{(0)} = \mathbf{0}$, for the linear system

$$\begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}.$$

6. (12%) Show that the vector $\mathbf{x}^*$ is a solution to the positive definite linear system $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}^*$ minimize

$$g(\mathbf{x}) = \frac{1}{2}\langle \mathbf{x}, A\mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{b} \rangle.$$

7. (12%) Show that a symmetric matrix A is positive definite if and only if all the eigenvalues of A are positive.

8. (12%) What does the Newton's method (for nonlinear system of equations) reduce to for the linear system $A\mathbf{x} = \mathbf{b}$.

9. (12%) Derive the formula for the Forward-Difference method for the parabolic partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 1, \quad t > 0,$$

subject to the conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

$$u(x,0) = f(x), \quad 0 \le x \le 1.$$

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