1. Let A, B and C be $n \times n$ matrices. Prove or disprove the following statements:

   (a) $\det(AB) = \det(BA)$; (5%)

   (b) trace $(AB) =$ trace (BA); (5%)

   (c) rank $(AB) =$ rank (BA); (5%)

   (d) If $Ax = 0$ has nonzero sulotion, then there exists b such that $Ax = b$ has no solution; (8%)

   (e) $A, A^2, \ldots, A^n$ and A^{n+1} are linearly independent; (8%)

   (f) If A is invertible, then A^{-1} can be represented by a polynomial of A; (8%)

   (g) If $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is invertible, then A and C are both invertible. (8%)

2. Let V be a finite-dimensional vector space.

   (a) Suppose W_1 and W_2 are subspaces of V such that $W_1 + W_2 = V$. Prove that $\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. (8%)

   (b) Show that if $T : V \to V$ is a linear transformation such that rank $(T) =$ rank (T^2), then $V =$ ken $(T) \oplus Im(T)$. (8%)

3. Let $A = \begin{bmatrix} \frac{5}{2} & -2 & 1 \\ -2 & \frac{5}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   (a) Find the characteristic polynomial and the minimal polynomial of T. (6%)

   (b) Determine the eigenvalues and eigenspaces of T. (6%)

   (c) Find an orthogonal matrix Q such that $Q^t AQ$ is diagonal. (6%)

4. Let V be a complex inner product space and T a linear operator on V. Prove that

   (a) if $\langle Tx, y \rangle = 0$ for all $x, y \in V$, then $T = 0$; (4%)

   (b) if $\langle Tx, x \rangle = 0$ for all $x \in V$, the $T = 0$; (5%)

   (c) if $|Tx| = |T^*x|$ for all $x \in V$, then T is normal; (5%)

   (d) if $|Tx| = |x|$ for all $x \in V$, then $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in V$. (5%)