Show all works

1. For the differential equation $\frac{dy}{dt} = -r(1-\frac{y}{T})(1-\frac{y}{K})y$, where $r > 0$ and $0 < T < K$.

(a) Find all the equilibrium solutions of the equation. [5%]

(b) Find the general solution of the equation. [5%]

2. Solve the differential equation $y'' + 6y' + 9y = 0$. [10%]

3. Solve the differential equation $mu'' + ku = F_0 \cos \omega t$, where m, k, F_0, and ω are constants,

(a) if $\omega_0 = \sqrt{\frac{k}{m}} \neq \omega$, [5%]

(b) if $\omega_0 = \sqrt{\frac{k}{m}} = \omega$. [5%]

4. Find the general solution of the differential equation $y''' - 6y'' + 12y' - 8y = 6e^t$. [10%]

5. Find the general solution in power series of $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$. [10%]

6. The Laplace transform of $f(t)$ is defined by $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$.

(a) Compute $\mathcal{L}\{t^n\}$, where n is a positive integer. [5%]

(b) Compute $\mathcal{L}\{t^{-\frac{1}{2}}\}$. [5%]

7. Find the general solution of the system $\mathbf{X}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$. [10%]

8. To approximate the solution $y = \phi(t)$ of the differential equation $\frac{dy}{dt} = f(t, y)$, we can use the improved Euler formula

$$y_{n+1} = y_n + \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})](t_{n+1} - t_n),$$

where y_{n+1} and y_n are approximation for $\phi(t_{n+1})$ and $\phi(t_n)$,respectively. Explain the geometric idea of this formula and draw a picture to illustrate your explanation. [10%]

9. Consider the system $\mathbf{X}' = A\mathbf{X}$, where A is a 2×2 constant matrix. Assume that A has complex eigenvalues $\lambda \pm i\mu$, where λ and μ are real. Show that the system can be transform into $\mathbf{Y}' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \mathbf{Y}$. [10%]

10. Consider the problem $y'' + \lambda y = 0$, $y(0) = 0$, $y'(L) = 0$. Show that if ϕ_m and ϕ_n are eigenfunctions, corresponding to the eigenvalues λ_m and λ_n, respectively, with $\lambda_m \neq \lambda_n$, then $\int_0^L \phi_m(x)\phi_n(x)dx = 0$. [10%]