1. Let X be an exponential random variable with parameter λ.

   (a) Find $P(|X - EX| \geq 2\sigma_X)$. (5%)

   (b) Prove that $P(\alpha \leq X \leq \alpha + \beta) \leq P(0 \leq X \leq \beta) \quad (\alpha, \beta > 0)$. (5%)

   (c) Is it possible that X satisfies the following relation?

$$P(X \leq 2) = 2P(2 < X \leq 4)$$

   If so, for what value of λ? (10%)

   Note: The density function of the exponential r.v. X is given by

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \ \lambda > 0, \\ 0, & \text{o.w.} \end{cases}$$

2. Let X be a non-negative random variable with distribution function F.

   Define

$$I(t) = \begin{cases} 1, & \text{if } X > t \\ 0, & \text{o.w.} \end{cases}$$

   (a) Prove that $\int_0^{+\infty} I(t)\,dt = X$. (5%)

   (b) Prove the $EX = \int_0^{+\infty} [1 - F(t)]\,dt$. (10%)

   (c) Prove that for $r > 0$, $EX^r = r\int_0^{+\infty} t^{r-1}[1 - F(t)]\,dt$. (5%)

   (d) Prove that $\sum_{k=1}^{+\infty} P\{X > k\} \leq EX \leq \sum_{k=0}^{+\infty} P\{X > k\}$. (5%)

3. Let X, Y be independent and geometrically distributed with parameter p, Geo (p).

   (a) Find $P(Y \geq X)$. (5%)

   (b) Find $P(X = Y)$ and $P(X > Y)$. (5%)

   (c) What is the distribution of $Z = \max\{X, Y\}$? (5%)

   Note. The density function of Geo (p) is given by

$$f(x;p) = \begin{cases} p(1-p)^x, & x = 0, 1, 2, \ldots, \ 0 < p < 1, \\ 0, & \text{o.w.} \end{cases}$$

(背面仍有題目,請繼續作答)

4. Let $(X, Y)'$ be a continuous random vector with joint density function

$$f(x,y) = \begin{cases} xe^{-x(1+y)}, & x > 0,\ y > 0 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the conditional density function of X, given that $Y = y \ (> 0)$. (10%)

(b) For $Y = y \ (> 0)$, find the conditional expectation, $E(X|y)$, and vairance, Var $(X|y)$. (5%)

5. Let X and Y be two independent uniform random variables over $(0, 1)$. Show that the random variables

$$U = \cos(2\pi X)\sqrt{-2\ln Y}$$

$$V = \sin(2\pi X)\sqrt{-2\ln Y}$$

are independent standard normal random variables. (15%)

6. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. r.v.'s with finite second moment. Let

$$Y_n = \frac{2}{n(n+1)}\sum_{i=1}^{n} iX_i.$$

Show that $Y_n \xrightarrow{p} EX_1$. (10%)