

## Ordinary Differential Equation

- (1) Solve the following differential equations: (20%)
- (a)  $\frac{dy}{dx} = \frac{2x+xy}{y^2+1}$
- (b)  $\frac{dy}{dx} = \frac{x+2y+2}{-2x+y}$
- (c)  $x^2 y'' = y'(3x - 2y')$
- (d)  $y'' - y' - 2y = 4x^2$

- (2) Study the asymptotic behavior of the differential equation (10%)

$$y'(x) = (y-1)(y-2)(y-3) \quad y(0) = c \in \mathbf{R}$$

as  $x \rightarrow \infty$  without solving the equation. You need to separate the range of  $c$ . Can you sketch the integral curve?

- (3) Use the method of variation of parameters to show that (20%)

$$y(x) = c_1 \cos x + c_2 \sin x + \int_0^x f(\xi) \sin(x-\xi) d\xi$$

is a general solution to the 2nd order differential equation

$$y'' + y = f(x)$$

where  $f(x)$  is a continuous function on  $(-\infty, \infty)$ .

- (4) Find the solution to the initial value problem (20%)

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{5t} \\ 2e^{2t} \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- (5) Apply the Laplace transform to solve the initial value problem (15%)

$$y'' + 2ty' - 4y = 1, \quad y(0) = y'(0) = 0$$

- (6) The Bessel function is defined by (15%)

$$J_\nu(x) \equiv \frac{1}{\pi} \int_0^\pi \cos(\nu t - x \sin t) dt \quad -\infty < x < \infty, \quad \nu \in \mathbf{R}$$

where  $f$  is a continuous and differentiable function. Show that when  $\nu$  is a integer then it satisfies the Bessel equation

$$x^2 J_\nu''(x) + x J_\nu'(x) + (x^2 - \nu^2) J_\nu(x) = 0$$