Show all works

1. Find the complete solution in the form $\mathbf{x}_p + \mathbf{x}_n$ to the system $\begin{cases} x+y+z=2 \\ x-y+z=2 \end{cases}$, where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_n$ is the general solution. [10%]

2. Can it be done for a 3×3 matrix A whose nullspace equals its column space. If you think it is right, then find one; otherwise, prove that it can not be done. (Nullspace = $\{\mathbf{x}:A\mathbf{x}=\vec{0}\}$ and Column space = {all linear combinations of column vectors of A}) [10%]

3. Find the Jordan form of $A = \begin{bmatrix} 4 & -2 & 0 & 2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & 0 & 6 \end{bmatrix}$ and the decomposition $A = MJM^{-1}$. [20%]

4. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ with three eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Show that the trace of A $= \lambda_1 + \lambda_2 + \lambda_3$. [20%]

5. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial of A. [5%]

(b) Find the minimal polynomial of A. [5%]

(c) Let $f(x) = x^5 - 7x^4 + 9x^3 + 9x^2 - 7x + 8$. Find $f(A)$. [5%]

(d) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. [5%]

6. Let $A^2 = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$. Find A. [10%]

7. One way to solve the system of simultaneous equations $\begin{cases} ax+by=\alpha \\ cx+dy=\beta \end{cases}$ is to solve its corresponding matrix equation,

$$A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

which tells us how the matrix A acts on the vector $\mathbf{x}$. Use this fact to show that

$$AB \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}.$$

[10%]