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We will use the following notations:

$\mathbb{R}$: the set of all real numbers.
$\mathbb{N}$: the set of all positive integers.
$\mathbf{sup}\,S$: the least upper bound of the set S such that $S \subseteq \mathbb{R}$.
$\ln(x)$: the natural logarithm function.

Please answer all questions (100%)

1. (20%)
   (a) (2%) State the definition of compact subset of $\mathbb{R}^n$.
   (b) (5%) Prove that every compact subset of $\mathbb{R}^n$ is closed and bounded.
   (c) (5%) Let K be a compact set in $\mathbb{R}$. Prove that there is an element $c \in K$ such that $c = \mathbf{sup}\{x \mid x \in K\}$.
   (d) (8%) Prove or disprove that every bounded sequence of elements in $\mathbb{R}$ has a convergent subsequence.

2. (20%) Let $g : [2, \infty) \longrightarrow \mathbb{R}$ be a function defined by $g(x) = \dfrac{x}{\ln(x)}$.
   (a) (4%) State the definition of uniformly continuous function.
   (b) (8%) Prove or disprove that g is uniformly continuous on $[2, \infty)$.
   (c) (8%) Let f be a real valued function defined on a bounded subset H of $\mathbb{R}$. Assume that f is uniformly continuous on H. Prove or disprove that the image of f is a bounded subset of $\mathbb{R}$.

3. (12%) Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $\varphi(x) = \displaystyle\int_0^{x+e^x} \sin(xt^2)\,dt$.
   (a) (5%) State the fundamental theorem of calculus.
   (b) (7%) Compute $\varphi'(0)$.

4. (10%) Let $f : [1, \infty) \longrightarrow \mathbb{R}$ be a function.
   (a) (2%) State the definition of convergence of the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$.
   (b) (8%)Prove or disprove that the improper integral $\displaystyle\int_1^{\infty} \frac{\sqrt{\ln(x)}}{x^2}\,dx$ is convergent.

5. (10%) For each $n \in \mathbb{N}$, let $f_n : [\frac{-1}{3}, \frac{1}{3}] \longrightarrow \mathbb{R}$ be a function defined by $f_n(x) = \sqrt{\dfrac{1-x^n}{1-x}}$.
   (a) (2%) State the definition of the uniform convergence of the sequence of functions $\{f_n\}_{n=1}^{\infty}$.
   (b) (8%) Prove or disprove that the sequence $\{f_n\}_{n=1}^{\infty}$ of functions converges uniformly on $[\frac{-1}{3}, \frac{1}{3}]$.

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6. (10%) Let $\{z_n\}_{n=1}^{\infty}$ be a bound sequence of real numbers.
   (a) (1%) State the definition of the lower limit $\liminf\limits_{n\to\infty} z_n$ of the sequence $\{z_n\}_{n=1}^{\infty}$.
   (b) (1%) State the definition of the upper limit $\limsup\limits_{n\to\infty} z_n$ of the sequence $\{z_n\}_{n=1}^{\infty}$.
   (c) (8%) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two bound sequences of real numbers. Prove that

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty} (x_n + y_n) \le \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

7. (18%) Let f and g be two real valued functions defined on a nonempty open subset D of $\mathbb{R}^n$. Assume that f, g and their partial derivatives are continuous on D. Let $S = \{x \in D \mid g(x) = 0\}$ and $c \in S$. Assume that $\nabla g(c) \neq 0$ and f takes its minimum value among all points of S at the point c. Prove that there is $\lambda \in \mathbb{R}$ such that $\nabla f(c) = \lambda \nabla g(c)$. (Here ∇h denote the gradient of the function h.)