※ 考生請注意：本試題 □可 ☑不可 使用計算機

Show all works

1. [15%] State the definition of vector space. Show that the set $\{a+b\sqrt{2} \quad a, b \text{ are rational numbers}\}$ together with the operations of addition and multiplication is a vector space. What is its dimension? How about $\{a+b\sqrt{2} \quad a, b \text{ are real numbers}\}$? What is its dimension? How about $\{a+b\sqrt{2} \cdot a, b \text{ are integers}\}$? What is its dimension? State your reason.

2. [10%] State the definition of unitary matrix. Let A and B be real matrices. If $A+iB$ is a unitary matrix, show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is an orthogonal matrix.

3. [10%] State the definition of Hermitian matrix. Let A and B be real matrices. If $A+iB$ is a Hermitian matrix, show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is a symmetric matrix.

4. [15%] Let A be a 2×2 matrix with entries of real numbers. Assume that A has complex eigenvalues $\lambda \pm i\mu$, where λ and μ are real. Find the matrix B such that $A = B^{-1}\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} B$. Under what conditions it is similar to a diagonal matrix? State your reason.

5. [20%] Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(**a**) Find the characteristic polynomial of A.

(**b**) Find the minimal polynomial of A.

(**c**) Let $f(x) = x^4 + 9x^3 + 9x^2 - 7x + 8$. Find $f(A)$.

(**d**) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

6. [10%] Find the Jordan form of $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$ and the decomposition $A = MJM^{-1}$

7. [10%] Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ with three eigenvalues λ_1, λ_2, and λ_3. Show that the matrix A has the trace $\lambda_1 + \lambda_2 + \lambda_3$.

8. [10%] Show that in R^3 the rotation around the unit vector $\mathbf{v} = (a,\ b,\ c)$ by angle ϕ is

$$Q = \cos\phi I + (1-\cos\phi)\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} - \sin\phi \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$