※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10%)

(a) Find the eigenvalues and eigenfunctions of the boundary-value problem
$$y'' + y' + \lambda y = 0, \ y(0) = 0, \ y(2) = 0$$

(b) Give an orthogonality relation for the eigenfunctions.

2. (10%)

Solve
$$y'' + y' + 2y = r(t), \ y(0) = 0, \ y'(0) = 0, \ r(t) = \begin{cases} 0 & \text{if} \ 0 < t < 1 \\ 1 & \text{if} \ 1 < t < 2 \\ 0 & \text{if} \quad t > 2 \end{cases}$$

3. (10%)

Write the general solutions of the given differential equations on $(0, \infty)$.

$(a) \ 16x^2 y'' + 16xy' + (16x^2 - 1)y = 0$
$(b) \ x^2 y'' + xy' + (9x^2 - 4)y = 0$

4. (10%)

Find the Fourier series of the following periodic function

$$f(x) = \begin{cases} 0 & \text{if} & -2<x<-1 \\ k & \text{if} & -1<x<1 \\ 0 & \text{if} & 1<x<2 \end{cases} \qquad f(x+4) = f(x)$$

5. (10%)

For the periodic function with a period of 4 in (a), its corresponding Fourier coefficients are shown in (b). Please plot the Fourier coefficients when the period is increased to 8 as shown in (c).

Waveform $f_L(x)$

(a)　2L = 4

Amplitude spectrum $a_n(w_n)$

$w_n = n\pi/L$

(b)

(c)　2L = 8
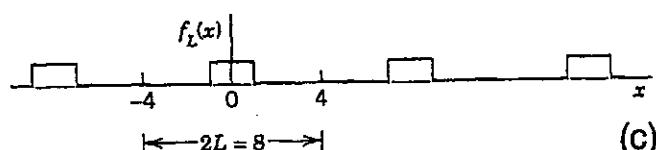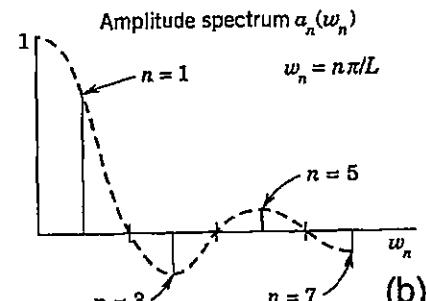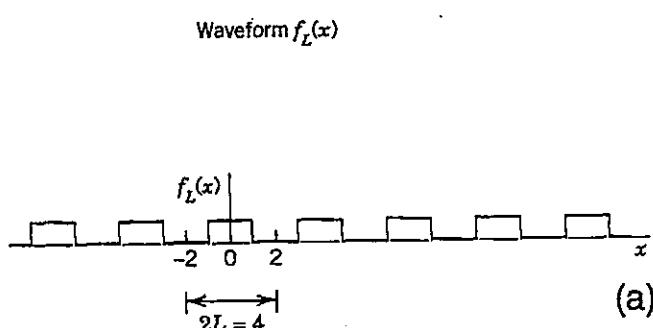
6. (10%)

   Find the work done by the force $\mathbf{F}(x,y,z) = 8xy^3z\mathbf{i} + 12x^2y^2z\mathbf{j} + 4x^2y^3\mathbf{k}$ acting along the path $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = -1$ to 1

7. (10%)

   Evaluate the outward flux $\iint_S (\mathbf{F} \cdot \mathbf{n})dS$ over the concentric spheres $S:$ $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $b > a$, where the vector field $\mathbf{F}(x,y,z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2}$

8. (10%)

   Solve the given initial-value problem

   $\dfrac{dx}{dt} = 2x + 4y$

   $\dfrac{dy}{dt} = -x + 6y$, where $\begin{matrix} x(0) = 5 \\ y(0) = 2 \end{matrix}$

9. (10%)

   Find values of a , b and c so that the given matrix is orthogonal.

   $$\begin{pmatrix} \dfrac{1}{\sqrt{2}} & a & \dfrac{1}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{2}} & b & \dfrac{1}{\sqrt{3}} \\ 0 & c & \dfrac{1}{\sqrt{3}} \end{pmatrix}$$

10. (10%)

    Evaluate the integral in the complex plane

    $\oint_C \dfrac{z^2 + 4}{z^2 - 5iz - 4} dz$, where C is the circle $\left| z - \dfrac{3}{2}i \right| = 2$