※ 考生請注意：本試題 □可 ☑不可 使用計算機

1. (8 %) (a) If complex function $f(z)$ has a simple pole at z = a on the real axis, please show the following theorem: $\lim_{r\to 0}\int_{C_2} f(z)\,dz = \pi i \operatorname{Res}_{z=a} f(z)$

, where $\operatorname{Res}_{z=a} f(z)$ is the residue of $f(z)$ at z = a, the path C_2 is

$C_2 : z = a + re^{i\theta}, 0 \le \theta \le \pi.$

(7 %) (b) Integrate the following complex function counterclockwise around C.

$\frac{\cos z}{z^n}$, n = 1, 2, ..., $C : |z| = 1.$

2. (8 %) (a) The convolution $f * g$ of functions f and g is defined by

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)g(p)\,dp.$$

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Please show the convolution theorem: $\mathscr{F}(f * g) = \sqrt{2\pi}\,\mathscr{F}(f)\mathscr{F}(g).$

(7 %) (b) Find the Fourier transform of $f(x)$.

$$f(x) = \begin{cases} k & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

3. (5 %) (a) Three matrices $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$, $B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$, and

$C = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$. Please point out and explain your answer which is the skew-Hermitian matrix and which is the unitary matrix.

(10 %) (b) Let $f = zy + yz$, $\vec{V} = [y, z, 4z - x]$. Please find (i) $\nabla^2(f^2)$ and (ii) $D_{\vec{V}} f$ at (3, 7, 5).

(5 %) (c) $\vec{F} = [z^2, x^3, y^2], C : x^2 + y^2 = 4, x + y + z = 0.$ Please evaluate the line integral $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.

(背面仍有題目,請繼續作答)

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4. Find the general solution of the following ODEs (ordinary differential equations):

   (a) $\dfrac{dy}{dx} = \cosh 4x$. (5%); (b) $\dfrac{dy}{dx} = \dfrac{4x^2 + y^2}{xy}$. (10%)

5. Consider the Cauchy problem for the equation

$$\frac{dy}{dx} = e^y + \cos x$$

   with the initial condition $y(0) = 0$. The solution has the form

$$y = Ax + Bx^2 + Cx^3 + \cdots.$$

   What are the A, B and C? (15%)

6. Solve the following PDEs (partial differential equations):

   (a) Show that $c^2\left(u_{rr} + \frac{1}{r}u_r\right) - u_{tt} = 0$ has solutions of the form

   $u(r,t) = \dfrac{V(r)}{r}\cos(nct)$, $n = 0,1,2,\ldots$. Find a differential equation for $V(r)$.

   (10%)

   (b) Solve the initial-value problem $u_x + 2u_y = 0$, $u(0, y) = 4e^{-2y}$. (10%)