[1] (A total of 20 points)

(a) (4 points) Write out the Maxwell's equations in a vacuum, in a differential form. Use the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ on the left side of the Maxwell's equations. The answers can be given either in SI units or Gaussian units, but need to be consistent.

(b) (4 points) Change the differential form of the Maxwell's equations into the integral form using the Gauss's divergence theorem

$$\int_S \mathbf{F} \cdot \mathbf{n} dS = \int_V (\nabla \cdot \mathbf{F})\, dV$$

which relates a surface integral and a volume integral, and the Stoke's theorem

$$\oint \mathbf{F} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

which relates a line integral and a surface integral. Here, $\mathbf{F}$ is an arbitrary vector, $\mathbf{dl}$, $dS$, and $dV$, are line, surface, and volume elements, respectively. The unit vector normal to the surface element $dS$ is given by $\mathbf{n}$.

(c) (4 points) Explain the physical meaning of each of the Maxwell's equations. Three out of the four relations have names. What are they called?

(d) (4 points) By introducing the vector potential $\mathbf{A}$ and the scalar potential $\Phi$, demonstrate that the differential form of Maxwell's equations can be reduced to a set of four equations, which is composed of one scalar equation and one vector equation.[1]

(e) (4 points) By introducing one of the gauge conditions (the Lorentz gauge condition), show that all the four equations in (d) can be expressed by the Poisson-type equations.

[1] A vector calculus identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$ can be used, where $\mathbf{F}$ is an arbitrary vector.

[2] (A total of 14 points)

(a) (4 points) When a conducting material is charged, the charge will appear uniformly on the surface of the conducting material. The charge inside will be zero. Explain this mechanism in terms of the Coulomb force. Hint: if only two electrons are given to a solid sphere of a conducting material, where do the two electrons reside?

(b) (5 points) After a lightning strikes a hollow copper tube, we found that the copper tube is compressed and distorted. This is due to the so-called pinch effect. Explain the mechanism of this pinch effect in terms of Ampere's law and the Lorentz force.

(c) (5 points) When space vehicles make a re-entry into the Earth's atmosphere, they suffer the communications blackout. The Earth's atmosphere is ionized at the re-entry. Explain the mechanism of the communications blackout in terms of index of refraction.

[3] (A total of 12 points)

(a) (5 points) The equation of motion for a charged particle in SI units is given by

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}.$$

Here, $m$ is the mass and $q$ is the charge of the particle. The position and the velocity of the particle are given by $\mathbf{x}$ and $\mathbf{v}$, respectively, and $\mathbf{B}$ is the magnetic field.

Solve the equation of motion for six components $(x, y, z)$ and $(v_x, v_y, v_z)$ in a three dimensional Cartesian coordinate system by taking the initial conditions $(x, y, z) = (x_0, y_0, z_0)$ and $(v_x, v_y, v_z) = (v_\perp, 0, v_\parallel)$ at time $t = 0$, and by assuming $\mathbf{B} = B_0\hat{z}$. Here, $B_0$ is a constant and $\hat{z}$ is the unit vector in the $z$ direction.

(b) (5 points) Now, consider the equation of motion in the presence of electric field $\mathbf{E}$,

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

by assuming $\mathbf{B} = B_0\hat{z}$ and $\mathbf{E} = E_y\hat{y}$ ($E_y$ is a constant and $\hat{y}$ is the unit vector in the $y$ direction). Take the same initial condition as in (a), and demonstrate that the motion in the $x-y$ plane can be decomposed into a cyclotron motion and a motion with a constant velocity in the $x$ direction (the latter is called the $\mathbf{E}\times\mathbf{B}$ guiding center motion).

(c) (2 points) In (a) and (b), we did not specify whether the particle is an ion or an electron. Compare the magnitude and state the direction of the $\mathbf{E}\times\mathbf{B}$ guiding center motion for a hydrogen ion and an electron, respectively. You can take the geometrical setting of (b).

[4] (A total of 24 points) This problem is to investigate the voltage across the RLC circuit in an analogy with damped and forced harmonic oscillators.

(a) (4 points) Consider a circuit as shown in Fig.4(a). The resistance, the inductance of coil, and the capacity of the capacitor are given by $R$, $L$, and $C$, respectively. Suppose that the charge on the capacitor is $Q$, and a current $I$ is flowing as in the figure, write out the Kirchhoff's voltage law.

(b) (4 points) Employing $I = -dQ/dt$ ($d/dt$ is the time derivative), write out a second order differential equation for $Q$. The equation should have the same form with damped harmonic oscillators. Applying initial conditions $Q = 0$ and $I = I_0$ at $t = 0$, solve the differential equation to obtain the form for $Q(t)$. Plot the solution as a function of time [draw a graph of $t$ versus $Q(t)$]. What is the cycle of the oscillation?

(c) (4 points) Now consider a case when a time dependent voltage $\Phi_0 \cos(\omega t)$ is applied, but with $R = 0$, as in Fig.4(b). Write out a second order differential equation for $Q(t)$. The equation should have the same form with forced harmonic oscillators. Applying initial conditions $Q = 0$ and $I = I_0$ at $t = 0$, solve the differential equation to obtain $Q(t)$. What happens if $\omega = 1/\sqrt{LC}$ is satisfied?

(d) (4 points) Finally, consider a case with $R \neq 0$ as in Fig.4(c). Write out a second order differential equation (ODE) for "$I(t)$". Here in (d), you do not need to solve the ODE to obtain an explicit form for $I(t)$. Instead, assuming a solution of the form $I = I_0 \cos(\omega t - \delta)$, obtain the amplitude $I_0$ and the phase difference $\delta$ in terms of $R$, $L$, $C$, $\omega$, and $\Phi_0$. What is the impedance of the circuit?

(e) (4 points) In (d), (i) the voltage across $R$, (ii) the voltage across $L$, (iii) the voltage across $C$, and (iv) the current $I(t)$, all oscillate in time with the frequency $\omega$. State the relative phase difference among the four. One way to do this is to draw a diagram of four vectors on a two dimensional space $(x, y)$ with $Tan^{-1}(y/x)$ being an angle of the phase.

(f) (4 points) Our RLC circuit analysis is based on quasi-static conditions which neglect one of the terms in the Maxwell's equations. Explain the idea of quasi-static current and point out which term in the Maxwell's equations is neglected.

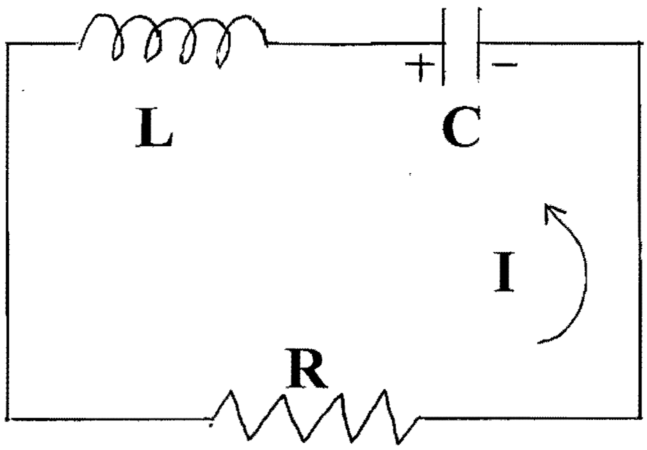


Fig.4(a)

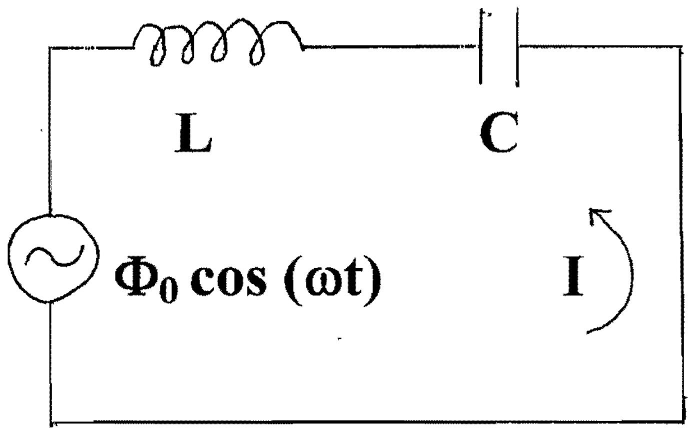


Fig.4(b)

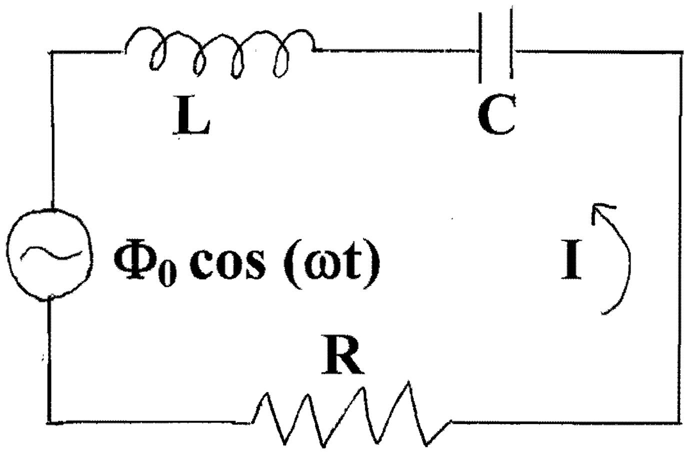


Fig.4(c)

[5] (A total of 14 points)

**(a)** (4 points) A test charge $q_T > 0$ is placed at the origin in a three dimensional spherical coordinate system. Using one of the Maxwell's equations, obtain the electric field $\mathbf{E}$ as a function of the radial coordinate $r$, the radial unit vector $\hat{r}$, and $q_T$. Then obtain the electrostatic potential $\Phi(r)$. Use a boundary condition $\Phi = 0$ at $r = \infty$.

**(b)** (3 points) In a plasma, due to the mobility of the electrons (electrons move much faster than the ions since $m_e \ll m_i$)[2] the net charge $q_T$ will be rapidly screened by the electrons. We would like to estimate the electrostatic potential as a function of the radial coordinate $r$. When a test charge $q_T > 0$ is placed again at $r = 0$, the Poisson equation in SI units is given by

$$\nabla^2 \Phi = \frac{1}{\varepsilon_0} e(n_e - n_i) - \frac{1}{\varepsilon_0} q_T \delta(r), \tag{1}$$

where $\varepsilon_0$ is the permittivity of free space, $e$ is the unit charge, $n_e$ and $n_i$ are the electron and ion density, respectively. Here, $\delta(r)$ is the Dirac delta function. From our knowledge of statistical mechanics, the electron density is given by the Boltzmann distribution

$$n_e = n_0 \exp\left(e\Phi/k_B T_e\right) \simeq n_0(1 + e\Phi/k_B T_e)$$

where $T_e$ is the electron thermal temperature ($e\Phi/k_B T_e \ll 1$ is assumed), and $k_B$ is called the Boltzmann constant. Here, the ion density can be regarded as constant; $n_i = n_0$, where $n_0$ corresponds to the equilibrium density. Show that away from $r = 0$, Eq.(1) becomes

$$\nabla^2 \Phi = \frac{n_0 e^2}{\varepsilon_0 k_B T_e} \Phi = \frac{\Phi}{\lambda_e^2}. \tag{2}$$

**(c)** (3 points) In the spherical coordinate, the Laplacian operator is given by

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right). \tag{3}$$

Assuming the solution of the form $\Phi(r) = f(r)/r$, show that $f(r) = C \exp(-r/\lambda_e)$, where $C$ is a constant to be determined. Finally, obtain an explicit form for $\Phi(r)$ which satisfies the solution of **(a)** very close to $r = 0$. What does $\lambda_e$ physically mean?

**(d)** (4 points) Prove that the total charge, when integrated from $r = 0$ to $r = \infty$, exactly cancels the test charge $q_T$.

[2] Here, $m_i$ and $m_e$ signify the mass of ions and electrons, respectively.

[6] (A total of 16 points)

(a) (3 points) Imagine the electric field component of an electromagnetic wave given by

$$E(x,t) = E_0 \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right],$$

where $\lambda > 0$ is the wave-length, $T > 0$ is the oscillation cycle, and $E_0$ is the amplitude of the wave. What is the phase velocity "$v_p$" of this wave in terms of $\lambda$ and $T$? The phase velocity is the speed of a constant wave height, for example how fast the position of the crest (or the trough) moves. Does the wave move to the right (positive $x$ direction) or to the left (negative $x$ direction)?

The electric field $E(x,t)$ can be also written in the form

$$E(x,t) = E_0 \sin(kx - \omega t).$$

Describe the relation between the wave vector "$k$" and $\lambda$, and the relation between the angular frequency "$\omega$" and $T$. What is the phase velocity in terms of $k$ and $\omega$?

(b) (4 points) Next, we consider a linear superposition of two waves, the first wave being

$$E_1(x,t) = E_0 \sin\left[(k + \Delta k)\, x - (\omega + \Delta\omega)\, t\right],$$

and the second wave being

$$E_2(x,t) = E_0 \sin\left[(k - \Delta k)\, x - (\omega - \Delta\omega)\, t\right].$$

Using a trigonometry formula

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

write out $E_s(x,t) = E_1(x,t) + E_2(x,t)$. As we discuss below, we are interested in the case where $k \gg \Delta k$ and $\omega \gg \Delta\omega$. To illustrate, taking $k = 1$ and $\Delta k = 0.1$, draw a graph of $x$ versus $E_s(x,0)$ within the domain $0 \le x \le 20\pi$.

(c) (3 points) The linear superposition in (b) demonstrates a beating of two waves whose $\omega$ and $k$ are only slightly different. Note that pure sine wave cannot convey any information. Only due to the beating between the two waves (or by modulation), we can convey information. The speed of the envelope (the slowly varying part) in $E_s(x,t)$ is called the group velocity "$v_g$". Taking the limit of $\Delta k \to 0$ and $\Delta\omega \to 0$, write out the group velocity in terms of $k$ and $\omega$.

(d) (3 points) If the electromagnetic wave is propagating in a vacuum, what are the phase velocity and the group velocity of the wave? Here, the speed of light is given by $c$.

(e) (3 points) In an unmagnetized plasma, the dispersion relation[3] of the electromagnetic wave is given by $\omega^2 = \omega_e^2 + k^2c^2$, where $\omega_e \ge 0$ is a constant called plasma frequency. What will be the phase velocity and the group velocity in the plasma? Can the group velocity exceed the phase velocity?

[3] The dispersion relation, in general, is the relation between the frequency $\omega$ and the wave vector $k$. Both $\omega$ and $k$ can be imaginary.

## Appendix

blackout 停電

communications blackout 無法通信

copper 銅

crest/trough 波峰/波谷

envelope 包絡線

index of refraction 折射率

solid sphere 球體

superposition 疊加