Show your steps clearly. Credit will be given for all the steps and derivations leading to the final results of the calculations.

1. Given a 2x2 matrix

$$\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c and d are real numbers, consider the matrix equation

$$\mathbf{Mx}=\lambda\mathbf{x},$$

where λ represents a constant, and $\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a 2-by-1 matrix of which the elements x_1 and x_2 are to be solved for.

(a) One can find non-trivial solutions for $\mathbf{x}$, that is, $\mathbf{x}\neq\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only for certain special values of λ. Find all those special values of λ in terms of a, b, c and d. (9%)

(b) For each of those special values of λ, find the corresponding $\mathbf{x}$ such that $x_1=1$. (6%)

2. (a) For the equation

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=0,$$

find solutions of the form $y=x^p$, where p is a constant. (6%)

(b) Find the most general form of the solution to the following second-order ordinary differential equation:

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=x^4\cos x \qquad (9\%)$$

3. (a) Find the leading behavior of

$$y(x)=\int_x^1 dt\ \sin(xt) \quad \text{as } x\to 0^+,$$

by giving the first three terms of the expansion. (6%)

(b) Find the first three terms of the asymptotic expansion of

$$y(x)=\int_x^\infty dt\sin(t^2) \quad \text{as } x\to\infty \qquad (9\%)$$

(背面仍有題目,請繼續作答)

4. Use Fourier transform to solve for $f(x,t)$ in the partial differential equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

along with initial condition $f(x,0)=\delta(x)$, where D is a positive constant and δ denotes the Dirac delta function. (20%)

5. Given $\mathbf{F}=(x+y^2+z^2)\hat{x}+(x^2+y+z^2)\hat{y}+(x^2+y^2+z)\hat{z}$ in Cartesian coordinates. Find the value of the surface integral $\int_S(\mathbf{F}\cdot\hat{\mathbf{n}})da$, where the surface S is a spherical surface with equation $x^2+y^2+(z-1)^2=1$ (i.e. spherical surface with center at $(x,y,z)=(0,0,1)$ and unit radius), $\hat{\mathbf{n}}$ is the unit vector normal to the surface and pointing outward, and da is the differential area. (15%)

6. For the probability distribution function

$$P(x)=\frac{1}{2\pi\lambda}\int_0^\infty \frac{d\sigma}{\sigma^2}\exp\left[-\frac{\ln^2(\sigma/\sigma_0)}{2\lambda^2}\right]\exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \text{for } -\infty < x < \infty,$$

where σ_0 and λ are positive constants,

(a) verify that $\int_{-\infty}^{\infty} P(x)dx = 1$; (5%)

(b) calculate the mean and the variance of the distribution. (15%)