國立成功大學109學年度碩士班招生考試試題

系　所：太空與電漿科學研究所

考試科目：應用數學

考試日期：0210，節次：2

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

[1] (A total of 24 points)

(a) (8 points)
A time dependent function $x(t)$ obeys an ordinary differential equation

$$\frac{dx(t)}{dt} + x(t) = A\cos(\omega t)$$

for $0 \le t$. Solve for $x(t)$. Initial condition is given by $x(0) = 0$. Here, A and ω are constants.

(b) (8 points)
A time dependent function $y(t)$ obeys an ordinary differential equation

$$\frac{dy(t)}{dt} + y(t) = A$$

for $0 \le t \le T$, and

$$\frac{dy(t)}{dt} + y(t) = 0$$

for $T \le t$. Solve for $y(t)$. Initial condition is given by $y(0) = 0$. Here, A is a constant. Note that the value of $y(t)$ should be continuous at $t = T$.

(c) (8 points)
Letting $T = 1/A$ in **(b)**, draw figures of t versus $y(t)$ within $0 \le t \le 5$, when (i) $A = 1$ and (ii) $A = 2$. Integrate

$$I = \int_0^\infty y(t)dt$$

and show I is independent of A (when $T = 1/A$).

[2] (10 points)

A complex number z satisfies

$$\exp(z^2) = i$$

where i is the imaginary unit. Find out the real and imaginary part of z.

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[3] (A total of 16 points)

(a) (8 points)
Prove that the matrix which rotate Cartesian coordinate variables (x, y, z) counterclockwise by angle ϕ_1 around the z-axis is given by

$$\mathbf{R}_1 = \begin{pmatrix} \cos(\phi_1) & -\sin(\phi_1) & 0 \\ \sin(\phi_1) & \cos(\phi_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (4 points)
Consider a matrix which rotate Cartesian coordinate variables (x, y, z) counterclockwise by angle ϕ_2 around the z-axis,

$$\mathbf{R}_2 = \begin{pmatrix} \cos(\phi_2) & -\sin(\phi_2) & 0 \\ \sin(\phi_2) & \cos(\phi_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that the matrix $\mathbf{R}_1\mathbf{R}_2$ and $\mathbf{R}_2\mathbf{R}_1$ rotate Cartesian coordinate variables (x, y, z) (counter-clockwise) by angle $\phi_1 + \phi_2$ around the z-axis.

(c) (4 points)
Write a 3×3 matrix which rotates Cartesian coordinate variables (x, y, z) by angle θ around the line $y = z$, $x = 0$.

[4] (A total of 18 points)

(a) (6 points)
Prove that the area of a circle of radius r is given by

$$S = \pi r^2$$

where $\pi = 3.141592...$ is the ratio of a circle's circumference to its diameter.

(b) (6 points)
Prove that the volume of a sphere of radius r is given by

$$V = \frac{4\pi}{3} r^3$$

(c) (6 points)
Describe one way to calculate the value of π.

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[5] (A total of 16 points)

(a) (4 points)
State the definition of even functions and odd functions. Prove that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)$$

for even functions and

$$\int_{-a}^{a} f(x)dx = 0 \cdot$$

for odd functions.

(b) (6 points)
State what Stokes' theorem is. Obtain a line integral

$$\oint \mathbf{A} \cdot \mathbf{dl}$$

for a vector $\mathbf{A} = \cos(x)\hat{\mathbf{i}} + x\,[1 - \sin(y)]\,\hat{\mathbf{j}}$ along a circle $x^2 + y^2 = 1$ on the xy-plane.

(c) (6 points)
State what Gauss' theorem is. Prove

$$\int_V \nabla \times \mathbf{B}dV = \int_S \hat{\mathbf{n}} \times \mathbf{B}dS$$

where $\int_V dV$ is a volume integral, $\int_S dS$ is a surface integral, and $\hat{\mathbf{n}}$ is a unit vector normal to the surface element dS. Here, $\mathbf{B}$ is a vector in general. Hint: You can use Gauss' theorem.

[6] (16 points)

Solve a two-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) = 0$$

for $\psi(x,y)$ within a domain $0 \le x \le a$ and $0 \le y \le b$. Apply boundary conditions

$$\psi(0,y) = \psi(a,y) = 0$$

$$\psi(x,0) = 0$$

$$\psi(x,b) = V$$

Here, a, b, and V are constants.