# 國立成功大學

# 110學年度碩士班招生考試試題

編　號：55

系　所：太空與電漿科學研究所

科　目：應用數學

日　期：0202

節　次：第 2 節

備　註：不可使用計算機

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

**Show your steps clearly. Except for Problem 2(a), credit will be given for all the steps and derivations leading to the final results of the calculations.**

1. Let $f$ be an arbitrary function, $f'(x) \equiv \frac{df(x)}{dx}$, and $f^{-1}$ be the inverse function of $f$. Derive the results of the following differentiations and integrations, with the answers expressed only in terms of $f$, $f'$, $f^{-1}$ and elementary functions: **(20%)**

   (a) $\frac{d[f^{-1}(x)]}{dx}$ (5%)

   (b) $\frac{d}{dx}\left(x^{f(x)}\right)$ (5%)

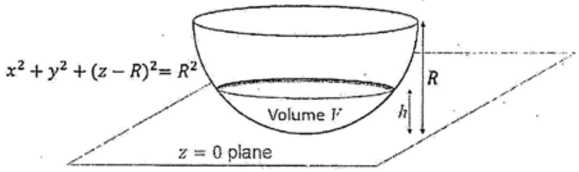
   (c) $\int^{x} dx \frac{f^{-1}(x)}{f'(f^{-1}(x))}$ (5%)

   (d) $\frac{d}{dy}\left[\int_{y}^{2y} dx f(x)\right]$ (5%)

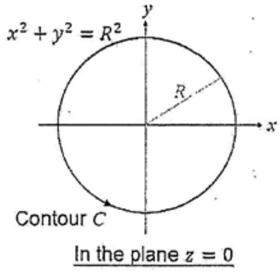
2. A water bowl, as shown in the figure below, is in the shape of a hemispherical (half sphere) surface of radius $R$. With the $+z$ direction defined to be vertically upward, the bowl is azimuthally symmetric about the $z$-axis. Hence, $R$ is also the height of the bowl. When the bottom of the bowl is placed on the $z=0$ plane, its surface corresponds to the equation $x^2+y^2+(z-R)^2=R^2$ for $0 \le z \le R$. **(15%)**

   (a) Without the need of derivation, give the volume of the water bowl. (3%)

   (b) When the bowl is being filled with water, the water volume, $V$, is a function of its height in the bowl, $h$. Set up an integral (either single, double or triple integrals) to represent the function $V(h)$, assuming that the thickness of the bowl is negligible. Then solve the integral to find $V(h)$. In particular, confirm that $V(R)$ is the volume of the bowl from Part (a). (12%)




3. Solve for the integral $I=\int_C \mathbf{F}\cdot d\boldsymbol{\ell}$, where $\mathbf{F}=xy^2\hat{x}+x^2y\hat{y}-z^2\hat{z}$ and $d\boldsymbol{\ell}$ goes counter-clockwise once along Contour $C$ as shown in the figure on the right, where $C$ corresponds to the circle with equations $\begin{cases} x^2+y^2=R^2 \\ z=0 \end{cases}$. **(15%)**



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4. Let $\mathbf{A}=\begin{pmatrix}3 & 0 & 1\\ 2 & 1 & 3\\ 1 & 0 & 3\end{pmatrix}$. Answer the following two parts of this problem related to $\mathbf{A}$. **(30%)**

   (a) Find all eigenvalues of $\mathbf{A}$ together with their respective eigenvector. (15%)

   (b) Consider the following system of differential equations:

$$\begin{cases}\dfrac{du}{dt}=3u+w\\ \dfrac{dv}{dt}=2u+v+3w\\ \dfrac{dw}{dt}=u+3w\end{cases} \tag{4.1}$$

   where $u$, $v$ and $w$ are functions of $t$, i.e. $u=u(t)$, $v=v(t)$ and $w=w(t)$. Notice that (4.1), the system of differential equations, can be expressed in matrix form:

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x},$$

   with $\mathbf{x}=\begin{pmatrix}u\\ v\\ w\end{pmatrix}$. Find the complete general solution for $u(t)$, $v(t)$ and $w(t)$ in the system. Show your derivation or explain the reasoning of your work for full credit. (15%)

5. Consider the equation

$$\frac{\partial^2 F}{\partial x^2}+4\frac{\partial^2 F}{\partial y^2}=0,$$

   where $F=F(x,y)$ is defined in the domain $\begin{cases}0\le x\le 1\\ 0\le y\le 1\end{cases}$. Using the method of separation of variables, solve for $F(x,y)$ under the following set of boundary conditions, which is also shown in the figure on the right:

$$F(0,y)=0;\quad F(x,0)=0;$$
$$F(1,y)=0;\quad F(x,1)=3\sin(\pi x)-\sin(3\pi x).$$

   **(20%)**