系所組別 太空天文與電漿科學研究所
考試科目 應用數學

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1. Find the following limits: (5 points)
   (a) $\lim_{x \to 0} x \ln x$
   (b) $\lim_{x \to 0} x \exp(1/x)$

2. Find the derivatives of the following functions (5 points)
   (a) $\frac{d}{dx} x^x$
   (b) $\frac{d}{dx}\left[\int_0^x dy F(x,y)\right]$, where $F(x,y)$ is a function of (x,y).

3. Find the following integrals (10 points)
   (a) $\int_{-\infty}^{\infty} dx F(x)\delta[a(x-b)]$, where $\delta(x)$ is delta function, $a(x-b)$ is the argument of the delta function, $F(x)$ is a function of x, and a and b are nonzero constants.
   (b) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$, using the contour integration in complex analysis.

4. Following equations are a set of coupled differential equations for $x(t)$ and $y(t)$,

$$\frac{dx}{dt} = -y,$$
$$\frac{dy}{dt} = x.$$

(20 points)

(a) Find the non-trivial general solutions, that is to say, excluding $x = y = 0$ solution.
(b) What is the shape of the trajectory of a particle that is described by $x(t)$ and $y(t)$ on the (x,y) plane?
(c) Find the constant of motion described by these equations.

5. Solve the following differential equation

$$\frac{\partial N}{\partial t} = -\alpha N^2,$$

where α is a positive constant. The initial condition is $N = N_0$ at $t = 0$. (15 points)

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6. The following is an equation for a vector $\boldsymbol{A}$ in Cartesian (x, y, z) coordinates

$$\nabla \times \boldsymbol{A} = \hat{z} + \frac{x}{L}\hat{y},$$

where L is a constant, and $\hat{y}$ and $\hat{z}$ are unit vectors in the y and the z direction respectively. Find a solution for $\boldsymbol{A}$ that is not a function of z, that is to say, $\partial \boldsymbol{A}/\partial z = 0$. (15 points)

7 The following is a differential equation

$$\frac{d^2x}{dt^2} + A\sin x = 0,$$

where A is a positive constant. (30 points)

(a) Find an analytic expression for dx/dt that satisfies the boundary condition that $dx/dt = 0$ at $x = x_0 \neq 0$.

(b) Find an analytic expression for the time it takes for a particle that is described by $x(t)$ to travel from $-x_0$ to x_0.