系所組別: 全校

考試科目: 微積分

※ 考生請注意:本試題 □可 ☑不可 使用計算機

1. Show that
$$\lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{\sqrt{n^2 - k^2}}{n^2} = \frac{\pi}{4}$$
   (Hint: Riemann sum) (10%)

2. Let
$$f(x) = x^x, \quad x > 0$$
   (a) Calculate $f'(x)$ (5%)
   (b) Find $\lim_{x \to 0^+} f(x)$ (5%)

3. Find $\int \frac{\sqrt{x}}{x^2 - 1} \, dx$ \quad (Hint: $x = u^2$) (10%)

4. Find $\int \cos(\ln x) \, dx$ \quad (Hint: integration by parts twice) (10%)

5. Find $\lim_{n \to \infty} \left(e^n \int_n^{2n} e^{-x^2} \, dx \right)$ \quad (Hint: pinching or sandwich theorem) (10%)

6. Find the radius of convergence of the power series (10%)
$$\sum \frac{3^k (k!)^3}{(3k)!} x^k$$

7. Use the method of Lagrange multiplier to find the extreme values of
$$f(x, y) = xy^3$$
   subject to the side condition $x^2 + 3y^2 = 16$. (10%)

8. Find the area of the surface by revolving the cycloid
$$x(t) = t - \sin t, \qquad y(t) = 1 - \cos t, \qquad 0 \le t \le 2\pi$$
   about the x-axis. (10%)

9. Let
$$g(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^4}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$
   Show that $\lim_{x \to (0,0)} f(x, y)$ does not exist. (10%)

10. Use polar coordinates to compute
$$\int_{\frac{1}{2}}^{1} \int_{1-x}^{x} \frac{1}{\sqrt{x^2 + y^2}} \, dy \, dx.$$
   (Hint: $\cos\theta + \sin\theta = \sqrt{2}\sin(\theta + \frac{\pi}{4})$) (10%)