國立成功大學七十五學年度轉學生考試(微積分 試題) 共 1 頁 第 1 頁

注意：不必抄題,但須標明題號.

一、求下列二極限：(1) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$; (16分)

(2) $\lim_{x\to 0^+}\frac{x^3\sin(1/x)}{x-[x]}$, 內 [ ] 為最大整數函數.

二、設 $f:[0,2]\longrightarrow\mathbb{R}$ 而 $f(x)=\begin{cases}x, & 若\ x\in[0,1),\\ 1, & 若\ x\in[1,2].\end{cases}$ (12分)

(1) 試問 f 是否為可積? 理由為何?

(2) 求一函數 $F:[0,2]\longrightarrow\mathbb{R}$ 使得 $\forall x\in[0,2]$, $F'(x)=f(x)$.

三、(1) 設 $a_n=\sum_{k=0}^{n-1}\frac{n}{4n^2+k^2}$. 試問序列 $\{a_n\}$ 是否收斂? (18分)

若是,則求其極限.

(2) 求冪級數 $\sum_{n=0}^{+\infty}\frac{(-1)^n(x-2)^{n+1}}{n+1}$ 之收斂區間.

四、(1) 求 $\int\frac{x\exp(x^2)}{\exp(x^2)+1}dx$ 之值; (2) 求 $\int_{1/2}^{\sqrt{3}/2}\sqrt{1-x^2}\,dx$ 之值; (24分)

(3) 證明瑕積分 $\int_0^{+\infty}\exp(-x^2)\,dx$ 收斂於 $\sqrt{\pi}/2$.

五、試繪製 $f(x)=\frac{1}{1+x^2}$ 之圖形. (14分)

六、(1) 設 $x=\rho\cos\theta\sin\phi$, $y=\rho\sin\theta\sin\phi$, $z=\rho\cos\phi$ 為球面坐標变換, 試求 $\frac{\partial\rho}{\partial x}(x,y,z)$ 及 (16分)

$$\begin{vmatrix}\frac{\partial x}{\partial\rho}(\rho,\theta,\phi) & \frac{\partial x}{\partial\theta}(\rho,\theta,\phi) & \frac{\partial x}{\partial\phi}(\rho,\theta,\phi)\\ \frac{\partial y}{\partial\rho}(\rho,\theta,\phi) & \frac{\partial y}{\partial\theta}(\rho,\theta,\phi) & \frac{\partial y}{\partial\phi}(\rho,\theta,\phi)\\ \frac{\partial z}{\partial\rho}(\rho,\theta,\phi) & \frac{\partial z}{\partial\theta}(\rho,\theta,\phi) & \frac{\partial z}{\partial\phi}(\rho,\theta,\phi)\end{vmatrix};$$

(2) 試利用重積分方法,求半徑為 r 之球体体積.

注意(甲)本試題分兩大部份. 第一部份(1—4題)每題10分. 第二部份(5—8題)每題15分. 務請依序作答, 否則酌予扣分.

(乙)不抄題, 但須標明題號.

第一部份

1.(a) 證: 若 $\lim_{x\to a} f(x) = A > 0$, 則必存在 a 為中心, δ 為半徑之區間, 在其內 $f(x)$ 恒為正. $f(a)$ 應如何規定?

(b) 問: 函數 f 應具備甚麼條件才可以作
$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)},\ n = 2, 3, 4, \cdots \text{ 之演算?}$$

2. 證明擺線(cycloid) $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$, 常數 $a > 0$, 恒凹向下(concave downward).

3. 設 (i) $f \in C^2$ (即 f'' 存在且連續)
(ii) $\Delta f(x) = f(x + \Delta x) - f(x)$
求 $\lim_{\Delta x \to 0} \frac{\Delta^2 f(x)}{(\Delta x)^2}$.

4. 證明 $\frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + \cdots + (n+1)^2 x^n + \cdots$, $|x| < 1$.

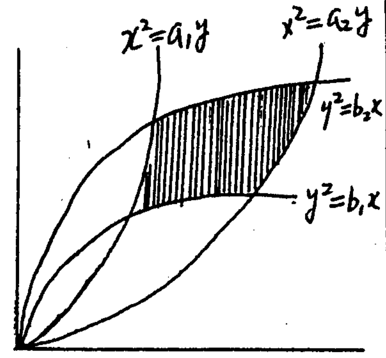
第二部份

5.(a) 證明 Cauchy-Schwarz 不等式
$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right),\quad a_k, b_k \text{ 為任意實數.}$$

(b) 設 $x+2y+3z+4=0$，求 $f(x,y,z)=x^2+y^2+z^2$ 之極小值.

6. (a) 求拋物線 $x^2=ay$，$y^2=bx$ $(a,b>0)$ 所圍區域之面積.

(b) 用(a)之結果，求：
$x^2=a_1y$，$x^2=a_2y$，$y^2=b_1x$，$y^2=b_2x$
$(a_2>a_1>0,\ b_2>b_1>0)$
四者所圍區域之面積.



7. (a) 設 $y^2(a-x)=x^3$，$a\neq 0$，求 $\int \frac{dx}{y}$.

(b) 求 $\int_0^1 \left[\int_x^1 \frac{\sin y}{y}\, dy \right] dx$.

8. $f(x,y)$ 經 $x=e^s$，$y=e^t$ 之代換變作 $g(s,t)$.
若 f 滿足 $x^2\frac{\partial^2 f}{\partial x^2}+y^2\frac{\partial^2 f}{\partial y^2}+x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0$，
g 滿足 $a\frac{\partial^2 g}{\partial s^2}+b\frac{\partial^2 g}{\partial t^2}+c\frac{\partial g}{\partial s}+d\frac{\partial g}{\partial t}=0$.
求 a. b. c. d 四常數的值.

注意：(1) 務請依序作答，否則酌予扣分.

(2) 不抄題，但須標明題號.

(3) 每道題均須寫出計算過程或說明理由，否則不予計分.

一. 求下列二極限：(a) $\lim_{x\to 0^+} x^{x^2}$

(b) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}$ (10分)

二(a) 試問級數 $\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$ 是否收斂？理由為何？

(b) $f(x)=x^{\sin x}$ 求 $f'(x)$. (10分)

三. 求下列積分值

(a) $\int_0^{\frac{\pi}{2}} e^x \sin x\,dx$ (b) $\int_{-1}^{1}\sin x^3\,dx$ (c) $\int_0^2\int_y^2 e^{x^2}\,dx\,dy$

(d) $\int\frac{dx}{\sqrt{1+e^x}}$ (20分)

四(a) 求曲線 $\begin{cases} x=t-\sin t \\ y=1-\cos t\end{cases}$ 在 $[0,2\pi]$ 間之弧長. (5分)

(10分)

(b) 設曲線C為 $z^2=x^2+y^2$ 及 $z=1+x+y$ 之交集，求C上距原點最近之點.

五. 設 $f(x)=\begin{cases} x^2\sin\frac{1}{x} & 若\ x\neq 0 \\ 0 & 若\ x=0\end{cases}$

(a) 問 f 在 $x=0$ 是否連續？理由為何？ (5分)

(b) 問 f 在 $x=0$ 是否可微？理由為何？ (10分)

六. 有一燈塔距海岸線3公里，以每分鐘8圈旋轉，試問當燈光與海岸線成45°角時，燈光沿海岸線移動之速度. (10分)

七. 若函数 f 與 g 在 $[a,b]$ 間連續且 $f(a) < g(a) < g(b) < f(b)$.

證明：存在 $p \in (a,b)$ 使得 $f(p)=g(p)$. (10分)

八. 以積分方法求橢圓 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 之面積 $(a>0,\ b>0)$. (10分)

７８學年度轉學生入學考試微積分試題

☆注意：
(1) 務請依序作答，否則酌予扣分。
(2) 不需抄題，但須標明題號。
(3) 每一題均須寫出計算過程或說明理由，只有答案不予計分。
(4) 試題中之 R 表實數系。

1. 試求 $\lim_{x\to 0}\left[\frac{1}{x^3}\right]\cdot x^5 = ?$ (其中 $[\cdot]$ 為最大整數函數) (10 分)

2. 設函數 $f: R\to R$, $p\in R$.
   (a) 試證：若 f 於點 p 為右可微(即右導數存在)，則 f 於點 p 為右連續. (7 分)
   (b) 若 f 於點 p 為右連續，試問 f 於點 p 是否必為右可微. (若為肯定，則證之，若為否定，則應舉一反例.) (8 分)

3. (a) 試證：$\forall x > 0$, $\ln(1 + 1/x) > (1 + x)^{-1}$； (7 分)
   (b) 試問函數 $f:(0,+\infty)\to R$ 而 $f(x) = (1 + 1/x)^x$ 於何處有極大、極小值. (8 分)

4. 設函數 $f: I\to R$，其中 I 為一非退化區間.
   (a) 試證：若 f 為連續，則 f 必具反導函數. (5 分)
   (b) 試問：若 f 具反導函數，f 是否必為連續？ (5 分)

5. 試求不定積分 $\int \frac{x^5}{(x^2 - 2)^{1/2}}\,dx$. (10 分)

6. 設 $f(x) = \int_{2x}^{x^3-4} \frac{x}{1+\sqrt{t}}\,dt$；試求 $f'(2) = ?$ (10 分)

7. 設 $f(x, y) = x^4 + y^4 - (x + y)^2$；試問 f 於何處有極大、極小值？ (15 分)

8. 試求積分 $\int_0^{+\infty} \exp(-x^2)\,dx = ?$ (15 分)

國立成功大學七十九學年度 轉學生考試（微積分 試題） 共 1 頁 第 1 頁

注意：(1) 務請依序作答，否則酌予扣分。

(2) 不抄題，但須標明題號。

(3) 每題均須寫出計算過程或說明理由，否則不予計分。

1. Find the following two limits and give the reasons for your each step:

$$\lim_{x \to 0} \frac{\cos(\sin x) + \sin(1 - \cos x) - 1}{x^2}; \qquad \lim_{x \to +\infty} \frac{\int_0^x \exp(t^2)\, dt}{x^2}.$$ 20%

2. Given that a and b are two real roots of the equation $f(x) = 0$, where $f(x)$ is a polynomoial in x, prove that there is at least one real root of the equation $f'(x) + f(x) = 0$ which lies between a and b. (HINT: Consider the function $g(x) = f(x)\exp x$.) 10%

3. (A) Prove that the improper integral $\int_0^{+\infty} t^{x-1} \exp(-t)\, dt$ is convergent for $x > 0$. 10%

(B) For $x > 0$, define $\Gamma(x) = \int_0^{+\infty} t^{x-1} \exp(-t)\, dt$. Prove that

$\Gamma(x + 1) = x\Gamma(x)$ and $\Gamma(n) = (n-1)!$,
where $(n-1)!$ is the factorial of $n-1$, for a positive integer n. 10%

4. (A) Is the following series convergent? If it is, also find its sum.

$$\frac{1}{1\cdot 3\cdot 5} + \frac{2}{3\cdot 5\cdot 7} + \frac{3}{5\cdot 7\cdot 9} + \frac{4}{7\cdot 9\cdot 11} + \cdots.$$
(HINT: Consider the rule for partial fractions.) 10%

(B) Find the interval of convergence of the power series

$$\sum_{n=1}^{+\infty} \frac{n+1}{(n+2)(n+3)} x^n.$$ 10%

5. Find the length of the arc of the parametric curve
$x = (\cos t) + (t\cdot\sin t)$, $y = (\sin t) - (t\cdot\cos t)$,
for $t \in [0, 2\pi]$. 10%

6. Determine the relative dimensions of a rectangular box, without a top, to be made from a given amount of material for the box to have the greatest possible volume. 10%

7. Find the total work done in moving an object in the counterclockwise direction once around the circle

$$x^2 + y^2 = a^2,$$
if the motion is caused by the force field

$$\vec{F}(x, y) = (\sin x - y)\vec{i} + (\exp y - x^2)\vec{j}.$$
Assume the arc is measured in meters and the force is measured in newtons. 10%

80 學年度轉學生入學考試微積分試題

☆注意：

(1) 務請依序作答，否則酌予扣分。
(2) 不需抄題，但須標明題號。
(3) 每一題均須寫出計算過程或說明理由，只有答案不予計分。
(4) 試題中之 $\mathbb{R}$ 表實數系。

1. (a) 令 $a_n > 0$ 且 $\sum_{n=1}^{\infty} a_n$ 收斂，證明 $\sum_{n=1}^{\infty} a_n{}^k$ 收斂.（此處 k 為大於 1 之常數）(8 分)

   (b) $\sum_{n=1}^{\infty} \sin^2(\frac{1}{n})$ 是否收斂 ？ 說明理由. (8 分)

2. 一登山者於週六早上四點登山，中午到達山頂過夜，隔天(週日)早上五點循原路下山，於早上十一點到達山腳出發點，證明該登山者在登山路線上某點於上、下山兩天中其手錶呈現同樣時間. (10 分)

3. 設函數 $f:\mathbb{R}\to\mathbb{R}$ 為連續且滿足方程式 $f(x)=\int_0^x f(t)\,dt+1$ 試求此函數 f. (10 分)

4. (a) 求 $\int_0^2\int_y^2 e^{x^2}\,dx\,dy$. (8 分)

   (b) 令 $R=\{(x,y): x^2+y^2\le 1\}$ 求 $\iint_R \sqrt{4x^2+4y^2+1}\,dA$. (8 分)

   (c) $S=\{(x,y): |x|+|y|\le 1\}$ 求 $\iint_S e^{x+y}\,dA$. (8 分)

5. 求冪級數 $\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^{n+1}}{n+1}$ 之收斂集. (10 分)

6. 求下列積分

   (a) $\int_{-\infty}^{\infty} e^{(x-e^x)}\,dx$. (b) $\int_0^8 \frac{1}{1+\sqrt[3]{x}}\,dx$. (10 分)

7. 求下列極限

   (a) $\lim_{x\to 0^+} x\ln x$. (b) $\lim_{x\to 0} x^{x^2}$. (10 分)

8. 用積分方法求橢圓 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 之面積 $(a>0,\ b>0)$. (10 分)

國立成功大學八十一學年度交管轉系生考試（微積分試題） 共 1 頁 第 1 頁

備註：此與二年級題相同

1. (i) Suppose $u = f(x,y,z)$, $z = g(x,y,t)$ and $y = h(x,t)$ are differentiable real-valued functions in suitable domains. Find $\frac{\partial u}{\partial x}(x,t) =?$ $\frac{\partial u}{\partial t}(x,t) =?$ 10%

   (ii) Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous and $g : \mathbf{R} \to \mathbf{R}$ is defined by $g(t) = \int_t^{t^2} \left(\int_0^{x^3} f(x,y)\, dy \right) dx$. Determine $g'(t) = ?$ 10%

2. (i) Is the integral $\int_{0+}^{1} \sin \frac{1}{x}\, dx$ convergent ? Justify your answer. 10%

   (ii) Let $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}$. Is $\{a_n\}$ convergent ? Justify your answer. 10%

3. (i) Find the work done by the force $\mathrm{F}(x,y,z) = (yz, xz, xy)$ in moving an object from $(0,0,0)$ to $(1,2,3)$ along the curve $\vec{\gamma}(t) = (t, 2t, 3t)$. 10%

   (ii) Use Green's Theorem to evaluate $\int_C x^2 y\, dx + 3xy\, dy$, where C is the positively oriented simple closed curve determined by the graphs of $y = x^2$ and $y = \sqrt{x}$. 10%

4. Let f be differentiable for $x > 0$. Prove or disprove

   (i) If $\lim_{x \to \infty} f(x) = 0$, then $\lim_{x \to \infty} f'(x) = 0$. 10%

   (ii) If $\lim_{x \to 0+} f(x) = \infty$, then $\lim_{x \to 0+} f'(x) = -\infty$. 10%

5. (i) Evaluate $\int_0^1 \left(\int_x^1 \frac{\sin y}{y}\, dy \right) dx$. 10%

   (ii) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\sqrt{3}}}$. 10%

1. (i) Suppose $u = f(x,y,z)$, $z = g(x,y,t)$ and $y = h(x,t)$ are differentiable real-valued functions in suitable domains. Find $\frac{\partial u}{\partial x}(x,t) =?$ $\frac{\partial u}{\partial t}(x,t) =?$ 10%

   (ii) Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous and $g : \mathbf{R} \to \mathbf{R}$ is defined by $g(t) = \int_t^{t^2} (\int_0^{x^3} f(x,y)\,dy)\,dx$. Determine $g'(t) = ?$ 10%

2. (i) Is the integral $\int_{0+}^{1} \sin\frac{1}{x}\,dx$ convergent ? Justify your answer. 10%

   (ii) Let $a_n = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}$. Is $\{a_n\}$ convergent ? Justify your answer. 10%

3. (i) Find the work done by the force $\mathrm{F}(x,y,z) = (yz, xz, xy)$ in moving an object from $(0,0,0)$ to $(1,2,3)$ along the curve $\overrightarrow{\gamma}(t) = (t, 2t, 3t)$. 10%

   (ii) Use Green's Theorem to evaluate $\int_C x^2y\,dx + 3xy\,dy$, where C is the positively oriented simple closed curve determined by the graphs of $y = x^2$ and $y = \sqrt{x}$. 10%

4. Let f be differentiable for $x > 0$. Prove or disprove

   (i) If $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f'(x) = 0$. 10%

   (ii) If $\lim_{x\to 0^+} f(x) = \infty$, then $\lim_{x\to 0^+} f'(x) = -\infty$. 10%

5. (i) Evaluate $\int_0^1 (\int_x^1 \frac{\sin y}{y}\,dy)\,dx$. 10%

   (ii) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^{\sqrt{3}}}$. 10%

國立成功大學八十二學年度轉學生考試(微積分試題) 共 1 頁 第 1 頁

注意：一、務請依序作答，否則酌予扣分.
二、不抄題，但須標明題號.
三、每題均須寫出計算過程或說明道理，否則不予計分.

1. (1) Let $f : [0, +\infty) \longrightarrow \mathbb{R}$ such that $f(x) = \cos\sqrt{x}$.
   (a) Find the derivative of f; 4%
   (b) Evaluate $\int \cos\sqrt{x}\, dx$. 4%

   (2) Find the limit $\lim_{x \to 0+} x^{\tan x}$. 8%

2. (1) Find the Maclaurin series of $f(x) = (1+x)^{\alpha}$, where $\alpha \in \mathbb{R}$. Show that the function $f(x)$ is analytic at $x = 0$. 8%

   (2) Find the 4-th order Taylor's expansion of $\sin(x + 2y)$ at the point $(0, 0)$. 8%

3. (1) Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$. 8%

   (2) Is the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

   continuously partially differentiable at $(0, 0)$. Is f differentiable at $(0, 0)$? 8%

4. (1) Show that $\int_0^{+\infty} e^{-x^2}\, dx$ is convergent. Also find the value which the improper integral converges to. 8%

   (2) Let R be the region between the graph of the curve $y = \exp(-x^2)$ and its asymptote. Find the volume of the solid generated by revolving the region R about the Y-axis. 8%

5. (1) Let $a_1 > 0$, $a_{n+1} = \frac{6(1+a_n)}{7+a_n}$. Show that the sequence $\{a_n\}$ is convergent and find its limit. 8%

   (2) Is the series $\sum_{n=1}^{+\infty} \frac{e^n n!}{n^n}$ convergent? If it is convergent, find its sum; otherwise, show the reason why it is divergent. 10%

6. (1) Let R be a connected compact region in $\mathbb{R}^2$ and let γ be the boundary of R such that it is a smooth closed oriented simple curve. Show that the area of R is

$$\frac{1}{2}\int_{\gamma} -y\, dx + x\, dy.$$

   8%

   (2) Consider the extrema of the function $f(x, y) = (y - x^2)(y - x^3)$. 10%

012

國立成功大學八十三學年度轉學生考試(微積分試題) 共 1 頁 第 1 頁

1. (a) Let $[x]$ denote the greatest integer function on $\mathbb{R}$. Show that $\lim_{x\to\infty} \frac{[x]^2}{x} = \infty$. 8%

   (b) Is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ convergent ? 7%

2. (a) Suppose $a > 0$. Evaluate $\int_0^a \frac{e^x}{e^x + e^{a-x}}\, dx$. 7%

   (b) Suppose $a, b > 0$ and R is the region $\left\{(x,y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$ in the plane. Evaluate $\iint_R \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy$. 8%

3. Find the limit as x approaches 0 of the ratio of the area of the triangle to the total shaded area in Figure 1. 10%

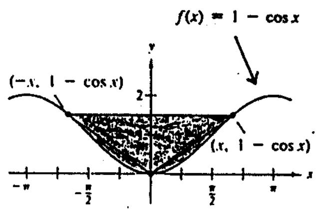


Figure 1

4. (a) Find the points on the paraboloid $z = 4x^2 + y^2$ at which the tangent plane is parallel to the plane $x + 2y + z = 6$. 9%

   (b) Suppose that a particle moving on a metal plate in the xy-plane has velocity $\vec{v} = \langle 1, -4 \rangle$ (cm/sec) at the point $(3, 2)$. If the temperature of the plate at points in the xy-plane is $T(x, y) = y^2 \ln x$, $(x \ge 1)$, in degrees Celsius, find $\frac{dT}{dt}$ at $(3, 2)$, where t denotes time. 8%

5. Suppose $f:[0,\infty) \to \mathbb{R}$: $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$.

   (a) Show that f is a one-to-one function. 5%

   (b) Let f^{-1} denote the inverse of f. Find $(f^{-1})'(0)$. 10%

6. Let Γ be the circle with radius 3 and center at the origin. A particle travels once around Γ in counterclockwise direction under the force field $\vec{F}(x, y) = \langle y^3, x^3 + 3xy^2 \rangle$. Use Green's theorem to find the work done by $\vec{F}$. 15%

7. In the plane let L be a line and Γ an ellipse that forms the boundary of a bounded region K. Use the intermediate-value theorem to show that there is a line parallel to L that cuts K into two pieces of equal area. 13%

注意事項：
1. 答案一律寫在試卷上，否則不予計分。
2. 請標明題號依序作答，不必抄題。
3. 試題應隨同試卷繳回，不得攜出試場。

1. Suppose $f:\mathbb{R}\to\mathbb{R}: f(x)=\int_1^x t^2\sqrt{3+t^4}\,dt+2$.
   (i) Determine where f is concave up or concave down. (4%)
   (ii) Show that f has the inverse function f^{-1}. (2%)
   (iii) Determine the range of f. (4%)
   (iv) Determine the equation of the tangent line to the graph of f^{-1} at $(2, f^{-1}(2))$. (6%)
   (v) Show that $\frac{4}{3} < f(0) < 2$. (4%)

2. Evaluate the following limits.
   (i) $\lim\limits_{x\to\infty}\frac{x-\sin x}{x}$. (5%)
   (ii) $\lim\limits_{k\to\infty}\sum\limits_{i=1}^{k}\frac{1}{\sqrt{k^2+i}}$. (5%)

3. (i) For any $x\in\mathbb{R}$, find the sum of the series $\sum\limits_{k=0}^{\infty}\frac{x^2}{(1+x^2)^k}$. (5%)
   (ii) Is the improper integral $\int_0^1 \frac{\ln x}{1-x^2}\,dx$ convergent? (10%)

4. Evaluate the following integrals.
   (i) $\int_0^1 \frac{1}{1+e^x}\,dx$. (7%)
   (ii) $\iint\limits_{\Omega}(x+y)^2\,d(x,y)$, where Ω is the parallelogram bounded by the lines $x+y=0$, $x+y=1$, $2x-y=0$ and $2x-y=3$. (8%)

5. A curve C in the plane is described by $\vec{\alpha}:[0,\pi]\to\mathbb{R}^2:\vec{\alpha}(t)=(-3\cos t, 2\sin t)$.
   (i) Find the area of the region enclosed by the curve C and the x-axis. (8%)
   (ii) Find the work done by the force field $F(x,y)=(y,-x)$ in moving an object from $(-3,0)$ to $(3,0)$ along C. (7%)

6. Let S be the circular paraboloid $x^2+y^2-z=1$. Use the gradient to find
   (i) the direction in which z increases most rapidly at $(1,2)$ and this maximum rate of increase, and (10%)
   (ii) the parametric equations of the normal line to S at $(1,2,4)$. (5%)

7. Suppose $x, y\in(0,1)$ and satisfy $x+y=1$. Use the technique of Calculus to show that $2^x+2^y<3$. (10%)