

注意：(1)務請依序作答，否則酌予扣分。

(2)不抄題，但須標明題號。

(3)每題均須寫出計算過程或說明道理，否則不予計分。

1. Find the following two limits and give the reasons for your each step:

$$\lim_{x \rightarrow 0} \frac{\cos(\sin x) + \sin(1 - \cos x) - 1}{x^2}; \quad \lim_{x \rightarrow +\infty} \frac{\int_0^x \exp(t^2) dt}{x^2} \quad 20\%$$

2. Given that a and b are two real roots of the equation $f(x) = 0$, where $f(x)$ is a polynomial in x , prove that there is at least one real root of the equation $f'(x) + f(x) = 0$ which lies between a and b . (HINT: Consider the function $g(x) = f(x)\exp x$.) 10%

3. (A) Prove that the improper integral $\int_0^{+\infty} t^{x-1} \exp(-t) dt$ is convergent for $x > 0$. 10%

- (B) For $x > 0$, define $\Gamma(x) = \int_0^{+\infty} t^{x-1} \exp(-t) dt$. Prove that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n) = (n-1)!$, where $(n-1)!$ is the factorial of $n-1$, for a positive integer n . 10%

4. (A) Is the following series convergent? If it is, also find its sum.

$$\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 5 \cdot 7} + \frac{1}{3 \cdot 5 \cdot 7 \cdot 9} + \frac{1}{4 \cdot 7 \cdot 9 \cdot 11} + \dots$$

(HINT: Consider the rule for partial fractions.) 10%

- (B) Find the interval of convergence of the power series

$$\sum_{n=1}^{+\infty} \frac{n+1}{(n+2)(n+3)} x^n. \quad 10\%$$

5. Find the length of the arc of the parametric curve $x = (\cos t) + (t \cdot \sin t)$, $y = (\sin t) - (t \cdot \cos t)$, for $t \in [0, 2\pi]$. 10%

6. Determine the relative dimensions of a rectangular box, without a top, to be made from a given amount of material for the box to have the greatest possible volume. 10%

7. Find the total work done in moving an object in the counterclockwise direction once around the circle

$$x^2 + y^2 = a^2,$$

if the motion is caused by the force field

$$\vec{F}(x, y) = (\sin x - y)\vec{i} + (\exp y - x^2)\vec{j}.$$

Assume the arc is measured in meters and the force is measured in newtons. 10%