

- 注意事項：1. 答案一律寫在試卷上，否則不予計分。  
2. 請標明題號依序作答，不必抄題。  
3. 試題應隨同試卷繳回，不得攜出試場。

1. (a) Evaluate the following limits.

(i)  $\lim_{x \rightarrow 0} \frac{1}{\lfloor \frac{1}{x} \rfloor} - x.$  (4%)

(ii)  $\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n \left( \frac{2i}{n} - 1 \right).$  (4%)

(b) Show that there exists a number between  $\frac{\pi}{2}$  and  $\pi$  such that  $\tan x = -x.$  (7%)

2. (a) Let  $f(x) = x^n e^{-x}$  for  $x \geq 0$  ( $n$  is a fixed but arbitrary positive integer). Show that the maximum value of  $f(x)$  is  $f(n).$  (5%)

(b) Show that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 - \frac{1}{n}\right)^{-n}. \quad (5\%)$$

3. Sketch the graph of  $f(x) = 2x^{\frac{1}{2}} + x^{\frac{3}{2}}$ . Indicate local extrema, inflection points, and concave structure. (10%)

4. (a) Let  $\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx, t > 0.$  Show that

$$\int_0^{\infty} x^{n-1} \exp(-x^2) dx = \frac{1}{2} \Gamma\left(\frac{n}{2}\right), \quad n \in N. \quad (5\%)$$

(b) Evaluate

$$\int_0^{\infty} e^{-x^2} dx. \quad (8\%)$$

(c) What is the value of  $\Gamma\left(\frac{1}{2}\right)?$  (2%)

5. (a) Show that

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \frac{\pi^2}{8}. \quad (2\%)$$

(b) Show that

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdots (2n+1)}, \quad n = 1, 2, 3, \dots \quad (4\%)$$

(c) What is the value of

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}} dx, \quad n = 1, 2, 3, \dots \quad (2\%)$$

(背面仍有題目, 請繼續作答)

(d) Use

$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^n, \quad |x| < 1,$$

to find a power series representation for  $\sin^{-1} x$ .

(6%)

(e) Show that

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

(4%)

(f) Use (a) and (e) to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(2%)

6. (a) Find the perimeter  $s$  of the cardioid with polar equation  $r = 1 + \cos \theta$ .

(7%)

(b) Find the surface area  $A$  generated by revolving the cardioid in (a) around the  $x$ -axis.

(8%)

7. (a) Evaluate

$$\int_0^2 \int_{\frac{1}{2}}^1 ye^{x^2} dx dy.$$

(5%)

(b) Find the volume of the solid region that is interior to both the sphere  $x^2 + y^2 + z^2 = 4$  of radius 2 and the cylinder  $(x-1)^2 + y^2 = 1$ .

(10%)